On the Linearization of Volterra Integral Equations

R. K. Miller*

Center for Dynamical Systems

Brown University

Providence, Rhode Island

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I. Introduction.

Given a nonlinear differential equation

$$x' = Cx + o(|x|), \quad (' = d/dt) \tag{1}$$

it is well known that the asymptotic stability of the linear system $y' = Cy$ implies the local asymptotic stability of the trivial solution of (1). All known proofs of this fact depend on the fact that solutions of the linear system decay exponentially or the equivalent fact that there exists a quadratic Lyapunov function for the linear system.

Consider a system of n equations of the form

$$x(t) = f(t) + \int_0^t a(t-s)g(x(s))ds, \quad t \geq 0 \tag{2}$$

where x, f and g are n-vectors, $a(t)$ is an $n \times n$ matrix and $g(0) = 0$. If f is "small" this system is often replaced by the more easily analyzed linear system

$$y(t) = f(t) + \int_0^t a(t-s)Jy(s)ds, \tag{3}$$

where J is the Jacobian matrix $g'(0) = (\partial g_i(0)/\partial x_j)$. However Levin and Nohel have proved by example that solutions of equations of the form (3) need not decay exponentially, c.f. [1, p.350, line (2.11)]. Therefore it has not been possible up to now to show that solutions of the linear system (3) approximate those of (2) in an appropriate sense, except in the case where solutions of (3) decay exponentially.

Nohel [9,10] has pointed out this gap in the theory of Volterra integral equations. He has asked whether or not a theory of linearization can be developed when solutions of (3) do not behave like solutions of an ordinary differential equation. The purpose of this paper is to provide in section II below a theory of linearization for equations (2) under very general assumptions on $a(t)J$. The essential tool in our analysis is Theorem 1 of [2] which we use in place of the usual estimates from ordinary differential equations.

The advantage of our method is that one can replace the local, nonlinear problem (2) by the linear equation (3) and the linear equation for its resolvent. These linear equations may be studied using known methods such as transform techniques. In Sections III, IV and V below we give some examples which illustrate this.

In the sequel we shall need the following notations and conventions. Let R^n denote real n-space with a norm $|x|$. Let $|D|$ denote the corresponding matrix norm. Let $BC[0,\infty)$ be the space of bounded continuous functions on $0 \leqq t < \infty$ with norm

$$\|h\|_0 = \sup \{|h(t)|; \quad 0 \leqq t < \infty\}.$$

Similarly $BC(R)$ will be the space of bounded continuous functions on $-\infty < t < \infty$ with norm

$$\|h\|_1 = \sup \{|h(t)|; \quad -\infty < t < \infty\}.$$

II. General Stability Conditions.

Concerning equation (2) we assume:

(A1) $a \in L^1(0,t)$ for each $t > 0$,

(A2) $f(t) \in BC[0,\infty)$,

(A3) $g(x) \in C^1(R^n), g(0) = 0$ and

(A4) the Jacobian matrix J is nonsingular.

Since we assume J is nonsingular, it is no loss of generality to assume J is the $n \times n$ identity matrix I. We need only replace $a(t)$ by $a(t)J$ and $g(x)$ by $J^{-1}g(x)$. Thus equation (3) may be rewritten in the form

$$(3') \qquad y(t) = f(t) + \int_0^t a(t-s)y(s)ds.$$

It is well known that the unique solution of equation (3') has the form

$$(4) \qquad y(t) = f(t) - \int_0^t b(t-s)f(s)ds, \quad (t \geqq 0)$$

where the matrix b is the resolvent kernel determined by the matrix equation

$$(5) \qquad b(t) = -a(t) + \int_0^t b(t-s)a(s)ds.$$

We assume that

(A5) the matrix b determined by (5) exists for all $t > 0$ and $|b(t)| \in L^1(0,\infty)$.

Theorem 1. If assumptions (A1-5) are satisfied then there exists $\epsilon_o > 0$ and $\epsilon_1 > 0$ such that when the solution $y(t)$ of (3') satisfies $\|y\|_0 \leq \epsilon_0$ the solution $x(t)$ of (2) exists for all $t \geq 0$ and $\|x\|_0 \leq \epsilon_1$.

Proof. Since $b \in L^1(0,\infty)$ it follows that equation (2) is equivalent to the system

$$x(t) = y(t) - \int_0^t b(t-s)G(x(s))ds, \tag{6}$$

where y is defined by line (4) and

$$G(x) = g(x) - x = o(|x|). \qquad (|x| \to 0)$$

Pick $\epsilon_1 > 0$ such that if $|x| \leq \epsilon_1$, then

$$2|G(x)| \int_0^\infty |b(s)|ds \leq |x|,$$

and $\int_0^\infty |b(s)|ds|g'(x) - I| < 1$. Pick $\epsilon_0 = \epsilon_1/2$. Let $Tx(t)$ be the function defined by the right hand side of equation (6). Let

$$S(0,\epsilon_1) = \{h \in BC[0,\infty); \ \|h\|_0 \leq \epsilon_1\}.$$

Our estimates on ϵ_0 and ϵ_1 imply that T is a contradiction map on $S(0,\epsilon_1)$. This proves Theorem 1.

Corollary 1. If (A1-5) are satisfied, then there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that when $\|f\|_0 \leq \epsilon_2$ the solution $x(t)$ of (2) exists for all $t \geq 0$ and satisfies $\|x\|_0 \leq \epsilon_1$.

Proof. Pick ϵ_2 such that

$$\epsilon_2(1 + \int_0^\infty |b(s)|ds) \leq \epsilon_1/2,$$

where ϵ_1 is the constant given in Theorem 1. Then equation (4) above implies $\|y\|_o \leqq \epsilon_o$. Thus Corollary 1 follows from Theorem 1 above.

Theorem 2. *Let (A1-5) hold and let ϵ_o and ϵ_1 be given by Theorem 1 above. If $\|y\|_o \leqq \epsilon_o$ and $y(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.*

Proof. Let Γ be the positive limit set of the solution $x(t)$, that is Γ is the smallest set such that $x(t)$ tends to Γ as $t \to \infty$. Since $x(t)$ is bounded it is easily shown that Γ is nonempty, compact and connected.

Since $x(t)$ solves equation (6), $y(t) \to 0$ and $b \in L^1(0,\infty)$ it follows from Theorem 1 of [2] that Γ is the union of solutions of

$$z(t) = - \int_{-\infty}^{t} b(t-s)G(z(s))ds, \tag{7.1}$$

$$|z(t)| \leq \epsilon_1. \qquad (-\infty < t < \infty) \tag{7.2}$$

Let $Tz(t)$ be the function defined by the right hand side of line (7.1) when $z \in BC(-\infty,\infty)$ and $\|z\|_1 \leq \epsilon_1$. The estimates in the proof of Theorem 1 above imply that T is a contraction map. Thus $z(t) \equiv 0$ is the unique solution of (7.1-2). This means that $\Gamma = \{0\}$. Thus $x(t) \to 0$ and the proof of Theorem 2 is complete.

Using Corollary 1 and Theorem 2 we obtain the following result.

Corollary 2. *Let* (A1-5) *hold and let* ϵ_1 *and* ϵ_2 *be given by Corollary* 1 *above. If* $\|f\|_0 \leq \epsilon_2$ *and* $y(t) \to 0$ *as* $t \to \infty$, *then* $x(t) \to 0$.

III. Applications: Integrable Kernels.

The purpose of this section is to apply the theory in Section II with the additional assumption that $a \in L^1(0,\infty)$. We shall need the following result.

Theorem 3 (Paley and Wiener). *Let* $a \in L^1(0,\infty)$. *Then the solution* b *of equation* (5) *is* $L^1(0,\infty)$ *if and only if the determinant*

$$\det\left(I - \int_0^\infty \exp(-st)a(t)dt\right) \neq 0, \tag{8}$$

in the right half plane $\operatorname{Re} s \geq 0$.

This theorem is proved by a trivial modification of the proof of Paley and Wiener of Theorem XVIII in [3, p. 60]. Paley and Wiener use Theorem 3 to study the asymptotic behavior of solutions of equation (3') in case $f(t) \to 0$ as $t \to \infty$. Their result has the following nonlinear generalization.

Theorem 4. Suppose (A1-4) *hold,* (8) *is satisfied for* $\operatorname{Re} s \geq 0$ *and* ϵ_2 *is given by Corollary* 1 *above. If* $\|f\|_0 \leq \epsilon_2$ *and* $f(t) \to 0$ *as* $t \to \infty$, *then* $x(t) \to 0$.

Proof. The solution of the linearized equation (3') is given by (4). Since $f(t) \to 0$ as $t \to \infty$ and $b \in L^1(0,\infty)$, the Lebesgue Dominated Convergence Theorem implies that $y(t) \to 0$. An application of Corollary 2 completes the proof of Theorem 4.

Levin [4] has obtained another nonlinear generalization of the Paley-Wiener result. His result is neither stronger nor weaker than Theorem 4 above. Levin studies a scalar equation $(n=1)$ while we allow $n > 1$. Our hypothesis on $a(t)$ is weaker than Levin's and our hypothesis on $g(x)$ stronger. Theorem 3 is a local result while Levin's result is global.

The condition $f(t) \to 0$ is essential to the proof of Theorem 4 above. If f has a different type of asymptotic behavior, it may still be possible to analyze the local behavior of solutions of equation (2). For example in Theorem 5 below, $f(t)$ is constant but not necessarily zero.

IV. Applications: Integrodifferential Equations.

The purpose of this section is to apply the theory of Section II to the study of the local behavior of integrodifferential equations of the form

$$x'(t) = mg(x(t)) + \int_0^t k(t-s)g(x(s))ds, \quad x(0) = x_0, \quad (t \geq 0) \tag{9}$$

where k is locally integrable and m is a constant. We allow $m = 0$. This system can be written in the form of equation (2) if one sets $f(t) \equiv x_0$ and

$$a(t) = m + \int_0^t k(s)ds.$$

We wish to investigate the asymptotic behavior of solutions of equation (9) when x_o is small. We remark that the definitions of stability and asymptotic stability of the trivial solution $x = 0$ of (9) are the same as for ordinary differential equations.

Theorem 5. Let f and a be as defined above. If (A3-4) hold, $a \in L^1(0,\infty)$ and (8) is true for $\operatorname{Re} s \geqq 0$, then for x_o sufficiently small

(i) the trivial solution of (9) is stable and

(ii) each solution of (9) tends to a constant as $t \to \infty$.

Proof. It follows from the proof of Corollary 1 above that for each ϵ, $0 < \epsilon < \epsilon_1$, there exists $\delta > 0$ such that $\|x\|_o \leqq \epsilon$ when $|x_o| \leq \delta$.

To prove part (ii) note that if $|x_o| \leq \epsilon_2$ then $|x(t)| \leq \epsilon_1$ for all $t \geqq 0$. Moreover

$$x(t) = (I-\int_o^t b(s)ds)x_o - \int_o^t b(t-s)G(x(s))ds.$$

Since $b \in L^1(0,\infty)$,

$$\lim_{t \to \infty} (I-\int_o^t b(s)ds)x_o = I - \int_o^\infty b(s)ds,$$

exists. By Theorem 1 of [2] the positive limit set of $x(t)$ is the union of solutions of

$$(10.1) \qquad z(t) = (I-\int_o^\infty b(s)ds)x_o - \int_{-\infty}^t b(t-s)G(z(s))ds,$$

$$(10.2) \qquad |z(t)| \leq \epsilon_1. \qquad\qquad (-\infty < t < \infty)$$

Let $S(0,\epsilon_1)$ be the closed sphere in $BC(R)$ with center at the origin and radius ϵ_1. Let S_o be the subset of $S(0,\epsilon_1)$

consisting of constant functions. The estimates on ϵ_1 in the proof of Theorem 1 imply that the right side of (10.1) defines a contraction map on $S(0,\epsilon_1)$ and on S_0. Therefore the unique solution of (10.1-2) is a constant function $z(t) \equiv z_0$. Thus the positive limit set of $x(t)$ is the single point z_0, $x(t) \to z_0$ as $t \to \infty$, and Theorem 5 is proved.

For x_0 small, the limit z_0 is obtained by solving the equation

$$z_0 = (I-\int_0^\infty b(s)ds)x_0 - (\int_0^\infty b(s)ds)G(z_0).$$

Let the solution be $z_0 = F(x_0)$. Then $F(0) = 0$ and F maps a neighborhood of $x_0 = 0$ diffeomorphically onto a neighborhood of $z_0 = 0$. This means that the trivial solution cannot be asymptotically stable.

V. Applications: A Reactor Problem.

The dynamic behavior of a continuous medium nuclear reactor can be described, under certain simplifying assumptions, by the following integrodifferential equations for the unknown u and T:

$$u'(t) = -\int_{-\infty}^{\infty} \alpha(x)T(x,t)dx, \tag{11.1}$$

$$T_t = T_{xx} + \eta(x)g(u(t)), \quad (-\infty < x < \infty,\ 0 < t < \infty) \tag{11.2}$$

with the initial conditions

$$u(0) = u_0, \quad T(x,0) = f(x). \quad (-\infty < x < \infty) \tag{12}$$

These equations have been extensively studied by Levin and Nohel, in the linear case $g(u) = u$ c.f. [1,5] and in the nonlinear case cf. [6]. In the reactor problem $g(u) = \exp(u) - 1$.

We wish to study the asymptotic behavior of solutions of (11) using the theory of Section II. Our analysis depends heavily on the papers [1,5, and 6] both for motivation and techniques. Since Levin and Nohel have treated the uniqueness problem for (11-12) we do not consider it further.

Let $*$ denote the L^2 Fourier transform. If f, α, and η are $L^2(R)$, then an elementary application of transform theory shows that $u(t)$ satisfies the equation

$$u'(t) = -\int_0^t m_1(t-s)g(u(s))ds - m_2(t), u(0) = u_0 \tag{13}$$

where for $j = 1,2$.

$$m_j(t) = (1/\pi) \int_0^\infty \exp(-x^2 t) h_j(x) dx,$$

and

$$h_1(x) = \text{Re } \eta^*(x)\alpha^*(-x), \; h_2(x) = \text{Re } f^*(x)\alpha^*(-x).$$

Using a Taubian theorem Levin and Nohel [1] study the linear equation

$$v'(t) = -\int_0^t m_1(t-s)v(s)ds - m_2(t), v(0) = v_0. \tag{14}$$

They prove

<u>Theorem 6</u> (Levin and Nohel). <u>Suppose</u> f, α <u>and</u> η <u>satisfy</u>:

(A6) $\quad f(x), \eta(x), \alpha(x) \in O(\exp(-\lambda|x|)), \lambda > 0, |x| \to \infty$

(A7) $\sup_{-\infty<x<\infty} \{|\alpha(x)|, \eta(x)|, |f(x)|\} < \infty.$

(A8) $h_1(x) \geq 0$ and $h_1(0) \neq 0.$

Then the solution $v(t)$ of (14) exists for all $t \geq 0$ and $v(t) = O(t^{-3/2})$ as $t \to \infty$.

Corollary 3. If the hypotheses of Theorem 6 are satisfied then there exists a positive constant K_1 (independent of v_o and f) such that for all $t \geq 0$

$$|v(t)| \leq K_1(|v_o|+\|f\|), \|f\| = \int_{-\infty}^{\infty} |f(x)|dx.$$

Proof. Let $v_1(t)$ be the solution of (14) when $v_o = 1$ and $m_2(t) \equiv 0$ and let $v_2(t)$ be the solution when $v_o = 0$. Then the general solution is $v_1(t)v_o + v_2(t)$. By Theorem 6 $v_1(t)$ is bounded.

Let V be the Laplace transform of v_2. Using lines 5.28 and 5.32 of [1] we see that for $-\infty < y < \infty$

$$V(iy) = H(y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-(iy)^{1/2}|x-s|)\alpha(x)f(s)dxds$$

where $H(y)$ is in $L^1(-\infty,\infty)$ and H depends only on α and η. Lemmas 5.1-5.6 of [1] show that V satisfies the hypotheses of Theorem 2 of [8, p. 266]. Therefore

$$|v_2(t)| \leq (2\pi)^{-1}\int_{-\infty}^{\infty} |H(y)|dy \int_{-\infty}^{\infty} |\alpha(x)|dx \int_{-\infty}^{\infty} |f(x)|dx.$$

This proves Corollary 3.

Using Theorem 2 and 6 we prove

Theorem 7. *Let* f, α *and* η *satisfy* (A6-8). *Let* g *satisfy* (A3) *with* $g'(0) = 1$. *Then there exist* $\delta > 0$ (*depending only on* η, α *and* g) *such that when* $|u_0| \leq \delta$ *and* $\|f\| \leq \delta$, *then the solution* $u(t)$ *of* (13) *exists for all* $t \geq 0$ *and* $u(t) \to 0$ *as* $t \to \infty$.

Proof. Let $b(t)$ be the resolvent kernel for equation (14), that is $b(t)$ solves (14) in the special case $v_0 = 0$ and $f = \eta$. By Theorem 6 we see that $v(t)$ and $b(t) = \mathcal{O}(t^{-3/2})$ as $t \to \infty$. Thus b is in $L^1(0,\infty)$. We know from Corollary 3 that $|v(t)|$ is small when $|u_0|$ and $\|f\|$ are small. An application of Theorem 2 completes the proof of Theorem 7.

Corollary 4. *Let the hypotheses of Theorem* 7 *hold. If* δ *is given by Theorem* 7 *and* $\|f\|, |u_0| \leq \delta$, *then* $u(t) \in L^1(0,\infty)$.

Proof. Fix u_0 and f with $|u_0|$ and $\|f\| \leq \delta$. Let $v(t)$ be the solution of (14). There exists $K > 0$ such that for all $t \geq 0$

$$|b(t)| \leq K(t+1)^{-3/2}, v(t)| \leq K(t+1)^{-3/2}.$$

Since $u(t) \to 0$, there exists $T > 0$ such that if $t \geq T$ then

$$|G(u(t))| = |g(u(t))-u(t)| \leq |u(t)|/(4K).$$

Let K_1 be a bound on $|G(u(t))|$ for $0 \leq t < \infty$. For all $t \geq 0$,

$$u(t+T) = v(t+T) - \int_0^T b(t+T-s)g(u(s))ds$$

$$- \int_0^t b(t-s)G(u(T+s))ds,$$

$$|u(t+T)| \leq K(t+T+1)^{-3/2} + KK_1 \int_0^T (t+T+1-s)^{-3/2}ds)$$

$$+ \int_0^t K(t+1-s)^{-3/2}|u(T+s)|/(4K)ds,$$

$$\leq K(t+T+1)^{-3/2} + 2KK_1((t+1)^{-1/2} - (t+T+1)^{-1/2})$$

$$+ \int_0^t (t+1-s)^{-3/2}|u(T+s)|/4ds$$

$$\leq H_1(t) + H_2(t) + \int_0^t H_3(t-s)|u(s+T)|ds.$$

The comparison theorem of Nohel [7, Theorem 2.1] implies that for $t \geq 0$, $|u(t+T)| \leq U(t)$, where U solves

$$U(t) = H_1(t) + H_2(t) + \int_0^t H_3(t-s)U(s)ds. \tag{15}$$

Since for any $t > 0$,

$$\int_0^t H_2(s)ds = 4KK_1(\sqrt{t+1} - \sqrt{t+T+1} - 1 + \sqrt{T+1}) \leq 4KK_1(\sqrt{T+1} - 1),$$

it follows that $H_2 \in L^1(0,\infty)$. Clearly H_1 and $H_3 \in L^1(0,\infty)$ and $\int_0^\infty H_3(s)ds \leq 1/2$. Thus the right hand side of equation (15) determines a contraction map on $L^1(0,\infty)$. Since $U(t)$ dominates $|u(t+T)|$, $u(t) \in L^1(0,\infty)$. This completes the proof of Corollary 3.

In order to study the asymptotic behavior of $T(x,t)$ we need the following additional assumption:

(A9) $f,\eta \in C(R)$ and η is locally Holder continuous.

Theorem 8. *Suppose* g *satisfies* (A3) *and* $g'(0) = 1$. *Let* f,α *and* η *satisfy* (A6-9). *Then for* u_0 *and* $\|f\|$ *sufficiently small problem* (11-12) *has a unique solution* $u(t),T(x,t)$. *Moreover,*

$$\sup_{-\infty<x<\infty} |T(x,t)| \to 0, \qquad (t \to \infty)$$

and $u(t) \to 0$ *as* $t \to \infty$ *with* $u \in L^1(0,\infty)$.

Proof. For u_0 and $\|f\|$ sufficiently small Theorem 7 and Corollary 3 imply the existence of a solution $u(t)$ of equation (13) such that $u \in L^1(0,\infty)$ and $u(t) \to 0$. Given this $u(t)$ define $T(x,t)$ on $-\infty < x < \infty$, $0 < t < \infty$ by

$$(16) \qquad T(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)\eta(y)g(u(s))dyds,$$

where $G(x,t) = (4\pi t)^{-1/2}\exp(-x^2/(4t))$. Using the same proof as in [7, p.264] we verify that the pair $u(t),T(x,t)$ is a solution of (11) and (12). Moreover, for any $t > 0$

$$|T(x,t)| \leq (4\pi t)^{-1/2}\int_{-\infty}^{\infty}|f(y)|dy+(4\pi)^{-1/2}\int_{-\infty}^{\infty}|\eta(y)|dy \int_0^t (t-s)^{-1/2}|g(u(s))|ds.$$

Since $g(u(t))$ is $L^1(0,\infty)$ it follows by dominated convergence that

$$\int_0^t s^{-1/2}|g(u(t-s))|\,ds = \int_0^t (t-s)^{-1/2}|g(u(s))|\,ds \to 0.$$

Therefore $T(x,t) \to 0$ as $t \to \infty$ uniformly for $-\infty < x < \infty$. This proves Theorem 8.

Theorem 8 is neither stronger nor weaker than the results in [6]. The advantage of Theorem 8 is that we avoid a hypothesis on the interconnection of f, α and η, c.f. [6, line 1.16]. The main disadvantage of Theorem 8 is that the result is local while the results of [6] are global.

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